

On fluid flow induced by a rotating magnetic field

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The interior of an insulating cylindrical container is supposed filled with an incompressible, electrically conducting, viscous fluid. An externally applied magnetic field is caused to rotate uniformly about an axis parallel to the cylinder generators (by applying two alternating components out of phase at right angles). Induced currents in the fluid give rise to a Lorentz force which drives a velocity field, which in general may have a steady and a fluctuating component. The particular case of a circular cylindrical container in a transverse magnetic field is studied in detail. Under certain reasonable assumptions, the resulting flow is shown to have only the steady component, and the distribution of this component is determined. Some conjectures are offered about the stability of this flow and about the corresponding flows in cavities of general shape.

1. Introduction

Progress in magnetohydrodynamics has been greatly hindered by the difficulties of achieving high magnetic Reynolds numbers in the laboratory. This number R_m is a dimensionless measure of the conductivity of the fluid and is defined by

$$R_m = 4\pi\mu\sigma LV, \quad (1.1)$$

where L and V are typical length and velocity scales of the moving fluid, and μ and σ its permeability and conductivity, respectively. The condition $R_m \gg 1$ is usually satisfied in astrophysical or geophysical problems on account of the large length scale involved, and many problems have been intensively studied by theoreticians for the limiting situation of infinite conductivity (or $R_m = \infty$). Laboratory experiments with mercury or liquid sodium are usually performed at $R_m \ll 1$, or at the highest speeds and largest length scales practicable, at $R_m \approx 1$. Consequently, it has not yet been possible to observe in the laboratory many of the effects predicted by theoreticians for the case $R_m \gg 1$, in particular those effects associated with the 'freezing' of the magnetic lines of force in the fluid.

The velocity V appearing in (1.1) must be interpreted as the relative velocity between the fluid and the magnetic field. It is difficult to obtain a high value of V in the laboratory simply by moving fluid relative to a *stationary* magnetic field. On the other hand, it should not be too difficult to obtain a large V by moving the field (by using suitable electrical windings outside the fluid) relative to the stationary fluid container. In particular a magnetic field \mathbf{B}_0 may be rotated relative to the fluid by applying two alternating components out of phase and at right angles.

A field of high frequency ω gives rise to a high relative velocity, $V = \omega L$, and this suggests that high effective values of $R_m (= 4\pi\mu\sigma L^2\omega)$ may in fact be accessible by quite easy means. This is no more than a restatement of the well-known fact that a conductor behaves almost like a perfect conductor when immersed in a high frequency magnetic field. Henceforth we shall assume

$$R_m = 4\pi\mu\sigma L^2\omega \gg 1. \quad (1.2)$$

If the fluid is contained inside an insulating container immersed in such a high frequency rotating field, then skin current flow, effectively excluding the field from the interior of the fluid. In the language of fluids, this skin is a magnetic boundary layer. Within this layer the curl of the Lorentz force generates vorticity, and a motion is transmitted to the core of fluid. The characteristic velocity of this fluid motion, U say, may be several orders of magnitude smaller than ωL . Indeed U will decrease as B_0 decreases so that circumstances certainly exist for which the magnetic Reynolds number defined in terms of U ,

$$\tilde{R}_m = 4\pi\mu\sigma LU, \quad (1.3)$$

is *small* compared with unity. This combination of conditions,

$$R_m \gg 1, \quad \tilde{R}_m \ll 1, \quad (1.4)$$

is the basis of the work described here; the latter condition may be expressed in terms of the amplitude of the applied magnetic field, when the resulting fluid motion is determined (see (2.21) and (2.22) below).

The significance of the condition $\tilde{R}_m \ll 1$ (which would in fact be hard to violate in the laboratory) is that the magnetic field distribution is unaffected by the motion of the fluid, and may be calculated *as though the fluid were a solid conductor* (It is of course far from uniform being zero in the core of the conductor; the distortion of the field is associated with the condition $R_m \gg 1$.) This is a striking simplification since the determination of the magnetic field everywhere is now a linear problem of well-known type (Landau & Lifshitz 1960). There remains, however, the formidable problem of solving the Navier–Stokes equations for the fluid motion, under a known distribution of Lorentz forces in the magnetic boundary layer. The type of motion that is produced will be largely governed in general by the magnitude of the Reynolds number $R = UL/\nu$, where ν is the kinematic viscosity of the fluid. The particular case of a circular cylinder in a rotating transverse field is greatly simplified by the fact that the streamlines turn out to be purely circular, and the inertia forces can be balanced by a radial pressure gradient; only in this case is the Reynolds number irrelevant.

2. Circular cylinder in a rotating transverse field

The magnetic field distribution

Suppose that the conducting fluid is contained inside the insulating boundary $r = a$ in cylindrical polars (r, θ, z) ; the region $r > a$ is supposed insulating. The magnetic field \mathbf{B}_0 is applied perpendicular to the cylinder axis, and is caused to rotate with uniform angular velocity ω , so that

$$B_{0r} \sim B_0 \cos(\theta - \omega t), \quad B_{0\theta} \sim -B_0 \sin(\theta - \omega t) \quad \text{as } r \rightarrow \infty. \quad (2.1)$$

The field and resulting flow are independent of z and are confined to the (r, θ) plane.

It is convenient to introduce the 'magnetic stream function' ψ_B , defined by

$$B_r = \frac{1}{r} \frac{\partial \psi_B}{\partial \theta}, \quad B_\theta = -\frac{\partial \psi_B}{\partial r}, \tag{2.2}$$

where from (2.1), $\psi_B \sim B_0 r \operatorname{Im} e^{i(\theta - \omega t)}$ as $r \rightarrow \infty$. (2.3)

We assume that
$$\psi_B = \left. \begin{aligned} f(r) e^{i(\theta - \omega t)} & \quad (r < a), \\ [B_0 r + g(r)] e^{i(\theta - \omega t)} & \quad (r > a), \end{aligned} \right\} \tag{2.4}$$

the imaginary part being understood. For $r > a$, $\nabla \wedge \mathbf{B} = 0$ so that $\nabla^2 \psi = 0$, i.e.

$$(\nabla^2 - 1/r^2) g(r) = 0,$$

i.e. $g(r) = C/r$ ($C = \text{const.}$), (2.5)

(the solution linear in r being rejected by virtue of (2.3)).

For $r < a$,
$$\partial \mathbf{B} / \partial t = \lambda \nabla^2 \mathbf{B}, \tag{2.6}$$

(neglecting the effect of the motion of the fluid by virtue of $R_m \ll 1$), where $\lambda = (4\pi\mu\sigma)^{-1}$ is the magnetic diffusivity of the fluid, i.e.

$$\partial B_r / \partial t = \lambda (\nabla^2 - r^{-2}) B_r,$$

or, using

$$\begin{aligned} B_r &= ir^{-1} f(r) e^{i(\theta - \omega t)}, \\ -i \frac{f(r)}{r} &= \lambda \left(\nabla^2 - \frac{2}{r^2} \right) \frac{f(r)}{r}, \end{aligned} \tag{2.7}$$

with general solution

$$f = Ar J_{\sqrt{2}}[k(1+i)r] + Br Y_{\sqrt{2}}[k(1+i)r], \tag{2.8}$$

where $k = (\omega/2\lambda)^{\frac{1}{2}}$. The part involving $Y_{\sqrt{2}}$ gives an infinite magnetic field at $r = 0$ and must be rejected, i.e. $B = 0$.

The constants A and C are determined by the conditions that both B_r and B_θ are continuous across $r = a$ (the possibility of surface currents being dismissed since σ is finite). These conditions give

$$A = \frac{2B_0}{2J_{\sqrt{2}}(p) + pJ'_{\sqrt{2}}(p)}, \quad C = \frac{-B_0 a^2 p J'_{\sqrt{2}}(p)}{2J_{\sqrt{2}}(p) + pJ'_{\sqrt{2}}(p)}, \tag{2.9}$$

where $p = k(1+i)a$. This completes the determination of the magnetic field distribution.

Now $R_m = 2(ka)^2$, so that for $R_m \gg 1$ the asymptotic formulae for the Bessel functions for large kr and p may be employed, e.g.

$$J_{\sqrt{2}}(p) \sim (2/\pi p)^{\frac{1}{2}} \cos [p - 2^{-\frac{1}{2}}\pi - \frac{1}{4}\pi], \quad J'_{\sqrt{2}} \sim -(2/\pi p)^{\frac{1}{2}} \sin [p - 2^{-\frac{1}{2}}\pi - \frac{1}{4}\pi].$$

After some reduction, the solution simplifies to the form

$$\begin{aligned} \psi_B &= B_0 \left\{ r - (a^2/r) \right\} \sin(\theta - \omega t) + (2^{\frac{1}{2}} B_0 a / kr) \sin(\theta - \omega t + \frac{1}{4}\pi) \quad (r > a), \\ &= (2^{\frac{1}{2}} B_0 / k) e^{-k(a-r)} \sin [(\theta - \omega t) + k(a-r) + \frac{1}{4}\pi] \quad (r > a). \end{aligned} \tag{2.10}$$

This form of the solution may alternatively be determined by first anticipating that for $R_m \gg 1$ the solution will have a boundary layer character, and that the terms $r^{-1} \partial/\partial r(f/r)$ and f/r^3 in (2.7) will be negligible compared with $\partial^2/\partial r^2(f/r)$ within the layer, but there is then a slight difficulty in eliminating the term corresponding to the $Y_{\sqrt{2}}$ -term that appears in the full general solution.

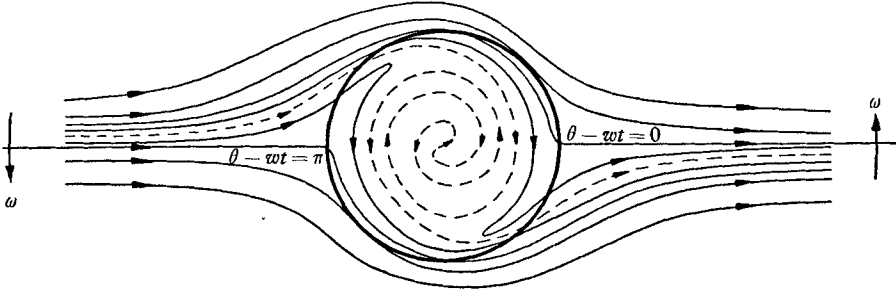


FIGURE 1. Lines of force $\psi_B = \text{const.}$, when a magnetic field rotates about a conducting cylinder.

The lines of force, $\psi_B = \text{const.}$, are sketched in figure 1. The first part of the solution (2.10), $B_0(r - a^2/r) \sin(\theta - \omega t)$, is the same as the stream function for potential flow past a cylinder; the remaining part of the solution (for $ka \gg 1$) is a small perturbation of this representing the penetration of the lines of force into the cylinder. The lines of force which start within a distance $O(k^{-1})$ of the axis $\theta' = \theta - \omega t = 0$ intersect the cylinder, intersecting first at $\theta' = \frac{1}{4}\pi$ and $\theta' = \frac{5}{4}\pi$. Those lines of force that intersect the cylinder surface within a distance $O(k^{-1})$ of the points $\theta' = \frac{3}{4}\pi, \frac{7}{4}\pi$ penetrate to near the centre of the cylinder, but the magnetic field is exponentially small inside the 'skin'. The tendency for the conductor to impede the motion of the lines of force across it is clearly marked. The skin thickness is

$$\delta = O(k^{-1}) = O(aR_m^{-\frac{1}{2}}). \quad (2.11)$$

The whole pattern of course rotates with angular velocity ω . The lines of force in figure 1 have been sketched for a moderate value of R_m in order to indicate the pattern. As $R_m \rightarrow \infty$, the pattern inside the cylinder shrinks into the skin, the dotted line of force being singular in the sense that it is the only line of force which penetrates to the centre in this limit.

The velocity distribution

The electric current in the fluid \mathbf{j} is purely in the axial direction, and from $4\pi\mu\mathbf{j} = \nabla \wedge \mathbf{B}$, is given by

$$4\pi\mu\mathbf{j} = -\nabla^2\psi_B \approx (\partial^2/\partial r^2)\psi_B, \quad (2.12)$$

making the thin layer approximation already employed above. Hence from (2.10),

$$4\pi\mu\mathbf{j} = 2^{\frac{3}{2}}B_0k e^{-k(a-r)} \cos \zeta, \quad (2.13)$$

where $\zeta = (\theta - \omega t) + k(a - r) + \frac{1}{4}\pi$. The rate of production of vorticity in the magnetic boundary layer by the Lorentz force $\mathbf{j} \wedge \mathbf{B}$ is

$$\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) = -\frac{1}{r} \frac{\partial(\psi_B, \mathbf{j})}{\partial(r, \theta)} \mathbf{k}, \quad (2.14)$$

where \mathbf{k} is a unit vector in the z -direction. Using (2.10) and (2.12), this gives

$$\begin{aligned} \nabla \wedge (\mathbf{j} \wedge \mathbf{B}) &= +B_0^2 k / \pi \mu r [-\sin \zeta (\sin \zeta - \cos \zeta) - \cos \zeta (\cos \zeta + \sin \zeta)] e^{-2k(a-r)} \\ &= -(B_0^2 k / \pi \mu r) e^{-2k(a-r)}. \end{aligned} \tag{2.15}$$

The rather surprising disappearance of the dependence on $(\theta - \omega t)$ is due to the fact that \mathbf{j} and ψ_B are exactly out of phase at every point. (This is certainly untrue if R_m is not large, in which case the flow would have a steady and a fluctuating component.)

Since the rate of production of vorticity in the skin depends only on r , the stream function ψ of the resulting velocity field likewise depends only on r , and the equation determining ψ (again using the thin layer approximation to replace r by a in the denominator of (2.15)) is

$$\nu \nabla^4 \psi = (B_0^2 k / \pi \mu \rho a) e^{-2k(a-r)}, \tag{2.16}$$

where ρ is the fluid density. The particular integral is

$$\psi_1 = (B_0^2 / 16 \pi \mu \rho \nu a k^3) e^{-2k(a-r)},$$

and the complementary function is

$$\psi_2 = A'r^2 + B' \log r.$$

The combination satisfying $\partial\psi/\partial r = 0$ on $r = a$ and $\partial\psi/\partial r$ finite at $r = 0$ is

$$\psi = \frac{B_0^2}{16 \pi \mu \rho \nu a k^3} \left[e^{-2k(a-r)} - \frac{kr^2}{a} \right]. \tag{2.17}$$

The velocity distribution is purely in the θ -direction and is

$$u_\theta = -\frac{\partial\psi}{\partial r} = \frac{-B_0^2}{8 \pi \mu \rho \nu a k^2} \left[e^{-2k(a-r)} - \frac{r}{a} \right]. \tag{2.18}$$

In the core region, defined by $k(a-r) \gg 1$, the velocity is the rigid body rotation

$$u_\theta \sim \Omega r \tag{2.19}$$

with angular velocity

$$\Omega = \frac{B_0^2}{8 \pi \mu \rho \nu a^2 k^2} = \frac{M^2}{R_m^2} \omega, \tag{2.20}$$

where

$$M = (\sigma / \rho \nu)^{\frac{1}{2}} B_0 a \tag{2.21}$$

is the Hartmann number as usually defined.

The relation between \tilde{R}_m defined in (1.3) and R_m is

$$\tilde{R}_m = \frac{\Omega a^2}{\lambda} = \left(\frac{M^2}{R_m^2} \right) \frac{\omega a^2}{\lambda} = \frac{M^2}{R_m}, \tag{2.22}$$

and the conditions (1.4) are therefore compatible as $R_m \rightarrow \infty$ keeping M constant. Possible orders of magnitude for mercury are indicated in table 1. Note that Ω decreases as ω increases (other things being constant) simply because the layer within which the driving Lorentz force is effective decreases in thickness, and the viscous reluctance of the fluid to respond becomes large. The smaller radius tends to favour a large angular velocity Ω in the fluid, which should be easily

observable in an experiment. The Reynolds number of the flow, though not relevant for this case of circular cross-section, is given as an indication of the importance of inertia forces for the flow inside other cross-sections.

3. The stability of the flow inside a circular cylinder

The flow described above has the property that the circulation decreases outwards within the layer between the rotating core and the cylinder which is at rest. Such a flow is prone to instabilities (Rayleigh 1916), and one might expect instabilities of the Taylor vortex type (Taylor 1923) when the rotation in the core exceeds some critical value. At high rotation rates, there would seem to be the possibility of a steadily rotating stable core bounded by a turbulent magnetic boundary layer.

a cm	ω sec ⁻¹	R_m	M	Ω sec ⁻¹	\tilde{R}_m	R
10	10 ⁵	10 ³	1	10 ⁻¹	10 ⁻³	10 ³
10	10 ⁶	10 ⁴	1	10 ⁻²	10 ⁻⁴	10 ²
1	10 ⁶	10 ²	1	10 ⁺²	10 ⁻²	10 ⁴

TABLE 1. Possible orders of magnitude in mercury

If a small perturbation is superposed on the steady flow determined above, the linearized equations describing the evolution of the perturbation are not influenced at all by the Lorentz force, whose only role is to determine the unperturbed state. The dimensionless number which appears in these equations, and which determines the onset of instability is the Taylor number,

$$T = (a\Omega^2\delta^3/\nu^2) = (a/\nu^2)(M^2\omega/R_m^2)^2(a/R_m^{\frac{1}{2}})^3,$$

i.e.

$$T = (\lambda/\nu)^2 M^4 R_m^{-\frac{7}{2}}. \quad (3.1)$$

Taylor (1923) found that the circulating flow between cylinders of nearly equal radii when the inner one rotates with angular velocity Ω is stable or unstable according as $T \lesseqgtr 1709$. The situation studied here is different only in that the velocity profile in the layer (2.18) and the inner boundary condition on the velocity perturbations are different; one might expect these differences to change the critical Taylor number by at most a factor of order unity. Hence it is probably safe to say that the criterion for stability of the flow studied here is

$$(\lambda/\nu)^2 M^4 R_m^{-\frac{7}{2}} < 10^3 \alpha, \quad (3.2)$$

where α is some number of order unity. For mercury, $\lambda/\nu \approx 10^6$, and the values in table 1 indicate that the first two situations are probably stable, while the third situation is probably unstable. It seems superfluous to attempt to make more precise stability calculations (by solving the stability eigenvalue problem) until there is some prospect of carrying out the experiment.

4. The flow inside non-circular containers

Consider now the case of a cavity which is cylindrical but not circular. This is equivalent to allowing a variation in the curvature $\kappa(\xi)$ of the cylinder surface, where ξ is the distance along the arc of the surface. It seems likely that the flow will be described by an equation like (2.16), allowing for inertia forces, and for the variation of curvature of the surface, viz.

$$-\frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} + \nu \nabla^4 \psi = \frac{B_0^2 k}{\pi \mu \rho} \kappa(\xi) e^{-2k\eta}, \quad (4.1)$$

where η is measured normally in from the surface. Built into this equation is the assumption, certainly valid for the circular case, that the only effect of the rotating magnetic field is to generate vorticity in the magnetic boundary layer at a rate constant in time, but proportional to the local curvature of the surface. There may in addition be a time dependent component of the flow of frequency 2ω , but for large ω one would expect this component to be of small amplitude (due to large viscous damping) and irrelevant to the determination of the mean flow.

If the Reynolds number R_s based on the skin thickness k^{-1} is small, the velocity component v in the ξ direction at the inner edge of the skin may be determined approximately from (4.1) in the form

$$v(\xi) = (B_0^2 k / \pi \mu \rho \nu) \kappa(\xi) / (2k)^3; \quad (4.2)$$

the condition for the consistency of this derivation is

$$R_s = (v/k\nu) = (M^2/R_m^{\frac{3}{2}}) (\lambda/\nu) \ll 1, \quad (4.3)$$

and if this is satisfied, the problem reduces to that of determining the flow when the tangential velocity at the inner edge of the skin is prescribed by (4.2). This problem further subdivides according as the Reynolds number based on the overall dimension $a = \kappa^{-1}$,

$$R = va/\nu \approx R_s R_m^{\frac{1}{2}}, \quad (4.4)$$

is large or small. If large, then there is a further boundary layer with closed streamlines of thickness $aR^{-\frac{1}{2}}$, and inside this a steady flow of uniform vorticity (Batchelor 1956). If small, then the non-linear terms in (4.1) are negligible everywhere, and the stream-function satisfies the biharmonic equation in the core.

If, on the other hand, R_s is large, but small compared with R_m , then a conventional boundary layer analysis would be needed to determine the solution of (4.1) through the skin. In this case R is necessarily large, and the flow in the core is of the uniform vorticity type.

The rate of production of vorticity in the skin is greatest where the curvature is greatest. An interesting extreme case is that in which the cross-section is a polygon, for example an equilateral triangle. The flow inside the triangle is that due to equal steady concentrations of vorticity production at the three corners. The rate of vorticity production along the straight edges where $\kappa(\xi) = 0$ is zero.

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